

Krivolinijski integral prve vrste (po luku)

Ako je c kriva data u ravni opisana jednačinom $y = \eta(x)$ gdje je $a \leq x \leq b$ tada

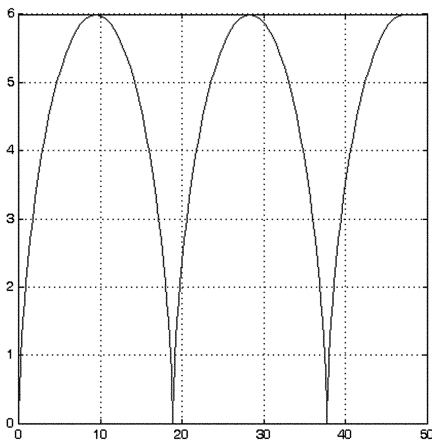
$$\int_c f(x, y) ds = \int_a^b f(x, \eta(x)) \underbrace{\sqrt{1 + (\eta'(x))^2}}_{ds} dx$$

Ako je c kriva opisana parametarskim jednačinama $x = \mu(t)$, $y = \eta(t)$ gdje je $t_1 \leq t \leq t_2$ tada

$$\int_c f(x, y) ds = \int_{t_1}^{t_2} f(\mu(t), \eta(t)) \underbrace{\sqrt{(\mu'(t))^2 + (\eta'(t))^2}}_{ds} dt$$

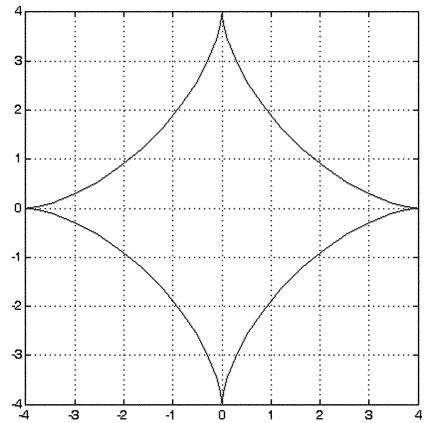
$$\left[\mu'(t) = \frac{\partial \mu}{\partial t} \right]$$

Krivolinijski integrali prve vrste f -ja triju promjenjivih $f(x, y, z)$ uzeti po prostornoj krivoj se računaju analogno. Krivolinijski integral prve vrste NE OVIŠI O SMJERU PUTA INTEGRACIJE.

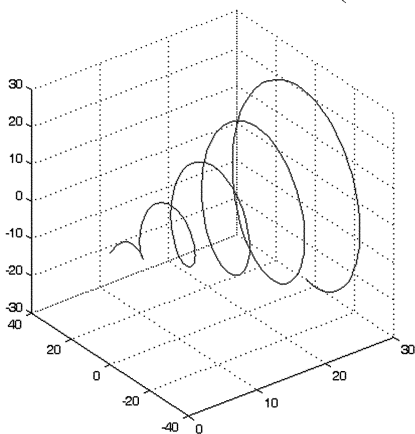


cikloida

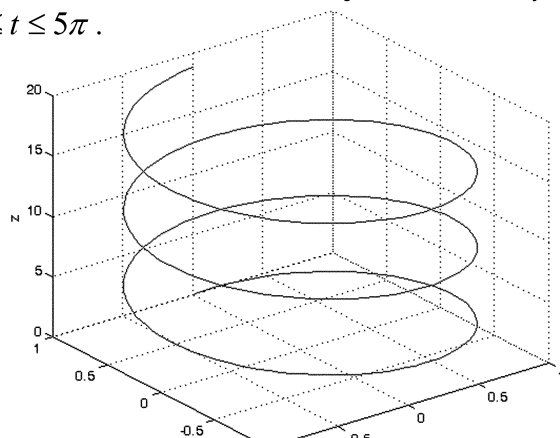
$$x = 3(t - \sin t), y = 3(1 - \cos t), 0 \leq t \leq 5\pi.$$



funkcija $x = 4 \cos^3 t, y = 4 \sin^3 t, 0 \leq t \leq 2\pi$.



funkcija $x = t, y = t \cos t, z = t \sin t, 0 \leq t \leq 30$.



zavojnica (spirala)

$$x = \sin t, y = \cos t, z = t, 0 \leq t \leq 6\pi.$$

Izračunati krivolinijski integral prve vrste

$$I = \oint_C \sqrt{x^2 + y^2} ds$$

gdje je C krug $x^2 + y^2 = ax$ ($a > 0$).

Rj. Prijetimo se

Ako je c kriva opisana parametarski $c: \begin{cases} x = \eta(t) \\ y = \mu(t) \\ t_1 \leq t \leq t_2 \end{cases}$ tada

$$\int_C f(x, y) ds = \int_{t_1}^{t_2} f(\eta(t), \mu(t)) \underbrace{\sqrt{(\eta'(t))^2 + (\mu'(t))^2}}_{ds} dt$$

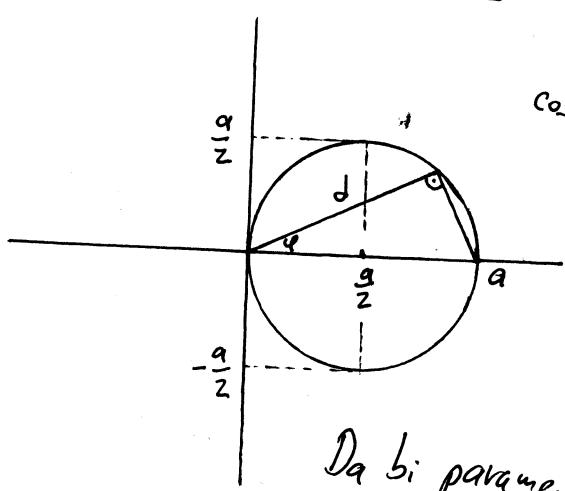
$$x^2 + y^2 = ax$$

$$x^2 - ax + y^2 = 0$$

$$x^2 - 2 \cdot x \cdot \frac{a}{2} + \frac{a^2}{4} + y^2 = \frac{a^2}{4}$$

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$

krug sa centrom $C\left(\frac{a}{2}, 0\right)$
poluprečnika $r = \frac{a}{2}$



$$\cos \varphi = \frac{d}{a}$$

$$d = a \cos \varphi$$

Da bi parametrizirali dati krug pomoći će nam polarne koordinate

$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

Kako r zavisi od ugla imamo

$$r = a \cos \varphi$$

$$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Parametrizacija datog kruga je

$$\left. \begin{aligned} x &= a \cos \varphi \cos \varphi = a \cos^2 \varphi \\ y &= a \cos \varphi \sin \varphi = \frac{a}{2} \sin 2\varphi \end{aligned} \right\} \Rightarrow x^2 + y^2 = a^2 \cos^2 \varphi$$

$$\left. \begin{aligned} x'_t &= 2a \cos \varphi (-\sin \varphi) = -2a \sin \varphi \cos \varphi = -a \sin 2\varphi \\ y'_t &= a \cos 2\varphi \end{aligned} \right\} \Rightarrow \sqrt{x_t'^2 + y_t'^2} = \sqrt{a^2 (\sin^2 2\varphi + \cos^2 2\varphi)} = a$$

$$I = \oint_C \sqrt{x^2 + y^2} ds = \int_{-\pi/2}^{\pi/2} a \cos \varphi \cdot a d\varphi = a^2 \int_{-\pi/2}^{\pi/2} \cos \varphi d\varphi = a^2 \sin \varphi \Big|_{-\pi/2}^{\pi/2} = 2a^2 \text{ traženo}$$

jer je

Izračunati krivolinijski integral $I = \int_L (4\sqrt[3]{x} - 3\sqrt{y}) dl$

između tački $E(-1; 0)$ i $F(0; 1)$

a) po pravoj EF ;

b) po liniji astroide $x = \cos^3 t, y = \sin^3 t$.

Rj. $I = \int_L (4\sqrt[3]{x} - 3\sqrt{y}) dl$

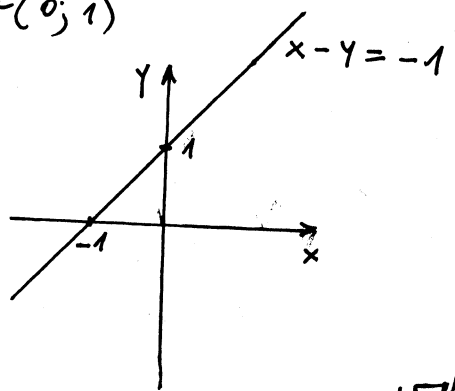
Ovo je krivolinijski integral prve vrste. Prisjetimo se
Ako je L kriva u ravni opisana jednačinom $y = \eta(x), a \leq x \leq b$ tada

$$\int_L f(x, y) dl = \int_a^b f(x, \eta(x)) \sqrt{1 + (\eta'(x))^2} dx$$

Ako je L opisana parametarskim jednačinama $\begin{cases} x = \mu(t) \\ y = \alpha(t) \end{cases}$ gdje $t_1 \leq t \leq t_2$

$$\int_L f(x, y) dl = \int_{t_1}^{t_2} f(\mu(t), \alpha(t)) \sqrt{(\mu'(t))^2 + (\alpha'(t))^2} dt$$

a) $E(-1; 0)$
 $F(0; 1)$

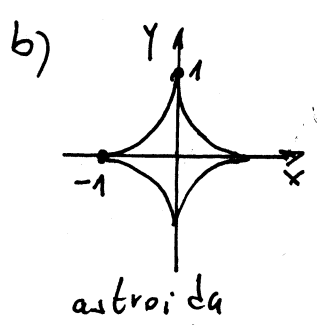


$-y = -x - 1, x \in [-1, 0]$ b) $y = x + 1$
 $y' = 1 \Rightarrow dl = \sqrt{1 + 1^2} dx = \sqrt{2} dx$

$$I = \int_L (4\sqrt[3]{x} - 3\sqrt{y}) dl = \int_{-1}^0 (4x^{\frac{1}{3}} - 3(x+1)^{\frac{1}{2}}) \sqrt{2} dx$$

$$= 4\sqrt{2} \int_{-1}^0 x^{\frac{1}{3}} dx - 3\sqrt{2} \int_{-1}^0 (x+1) dx =$$

$$= 4\sqrt{2} \cdot \frac{3}{4} x^{\frac{4}{3}} \Big|_{-1}^0 - 3\sqrt{2} \int_{-1}^0 (x+1)^{\frac{1}{2}} d(x+1) = 3\sqrt{2} (0 - 1) - 3\sqrt{2} \cdot \frac{2}{3} (x+1)^{\frac{3}{2}} \Big|_{-1}^0 = -5\sqrt{2}$$



$x = \cos^3 t, x' = -3\cos^2 t \sin t$
 $y = \sin^3 t, y' = 3\sin^2 t \cos t$

$$dl = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} dt$$

↑
taženo
rešenje

$$\sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t} = 3 \sqrt{\cos^2 t \sin^2 t (\cos^2 t + \sin^2 t)} =$$

$$= 3 |\sin t \cos t|$$

U našem slučaju t uzima vrijednost od $\frac{\pi}{2}$ do π , pa je

$$dl = -3 \sin t \cos t dt$$

$$l = \int_L (4\sqrt[3]{x} - 3\sqrt{y}) dl = \int_{\frac{\pi}{2}}^{\pi} (4\sqrt[3]{\cos^3 t} - 3\sqrt{\sin^3 t}) (-3 \sin t \cos t) dt$$

$$= -12 \int_{\frac{\pi}{2}}^{\pi} \cos^2 t \sin t dt + 9 \int_{\frac{\pi}{2}}^{\pi} \sin^{\frac{5}{2}} t \cos t dt =$$

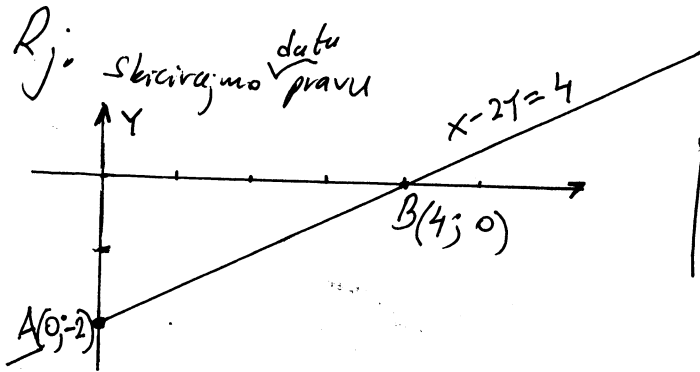
$$= +12 \int_{\frac{\pi}{2}}^{\pi} \cos^2 t d\cos t + 9 \int_{\frac{\pi}{2}}^{\pi} \sin^{\frac{5}{2}} t d\sin t = 12 \cdot \frac{\cos^3 t}{3} \Big|_{\frac{\pi}{2}}^{\pi} + 9 \cdot \frac{\sin^{\frac{7}{2}} t}{\frac{7}{2}} \Big|_{\frac{\pi}{2}}^{\pi}$$

$$= 4((-1)^3 - 0) + \frac{18}{7} (0 - 1^{\frac{7}{2}}) = -4 - \frac{18}{7} = -\frac{46}{7}$$

traženo
rješenje

Izračunati krivolinijski integral $\int_{AB} \frac{dl}{\sqrt{x^2+y^2}}$ po

odsečku prave $x-2y=4$ od tačke $A(0;-2)$ do tačke $B(4;0)$.



Prizetmo se kako se računa krivolinijski integral prvog tipa, ako je kriva integracije ^{curvni} opisana formulom $y=\eta(x)$, $a \leq x \leq b$

$$\int_c^d f(x,y) dl = \int_a^b f(x, \eta(x)) \sqrt{1+(\eta'(x))^2} dx$$

I način:

$x-2y=4$
 $2y=x-4$
 $y=\frac{1}{2}x-2$
 $y'=\frac{1}{2}$

$$\int_{AB} \frac{dl}{\sqrt{x^2+y^2}} = \int_0^4 \frac{\sqrt{1+\frac{1}{4}}}{\sqrt{x^2+(\frac{1}{2}x-2)^2}} dx = \frac{\sqrt{5}}{2} \int_0^4 \frac{dx}{\sqrt{\frac{5x^2}{4}-2x+4}}$$

$$= \frac{\sqrt{5}}{2} \cdot \frac{1}{\sqrt{\frac{5}{4}}} \int_0^4 \frac{dx}{\sqrt{x^2-\frac{8}{5}x+\frac{16}{5}}} = \left| x^2-\frac{8}{5}x+\frac{16}{5} \right|_0^4 = x^2-2 \cdot x \cdot \frac{4}{5} + \frac{16}{25} = (x-\frac{4}{5})^2 + \frac{64}{25}$$

$$= \int_0^4 \frac{d(x-\frac{4}{5})}{\sqrt{(x-\frac{4}{5})^2 + \frac{64}{25}}} = \ln \left| x-\frac{4}{5} + \sqrt{(x-\frac{4}{5})^2 + \frac{64}{25}} \right|_0^4 = \ln \left(\frac{16}{5} + \sqrt{\frac{16(16+4)}{25}} \right) - \ln \left(-\frac{4}{5} + \sqrt{\frac{16+64}{25}} \right)$$

$$= \ln \frac{\frac{16+8\sqrt{5}}{5}}{\frac{-4+4\sqrt{5}}{5}} = \ln \frac{4+2\sqrt{5}}{\sqrt{5}-1} \cdot \frac{(\sqrt{5}+1)}{(\sqrt{5}+1)} = \ln \frac{4+6\sqrt{5}+10}{5-1} = \ln \frac{14+6\sqrt{5}}{4}$$

II način

$x-2y=4$
 $x=2y+4$
 $\frac{\partial x}{\partial y} = 2$

$$\int_{AB} \frac{dl}{\sqrt{x^2+y^2}} = \int_{-2}^0 \frac{\sqrt{1+4}}{\sqrt{(2y+4)^2+y^2}} dy = \sqrt{5} \int_{-2}^0 \frac{dy}{\sqrt{5y^2+16y+16}} = \dots$$

ZAVRŠITI ZA VJEŽBU

⊕ Izračunati krivoliniski integral $\int_L (x-y) ds$ po kružnoj liniji $x^2 + y^2 = ax$.

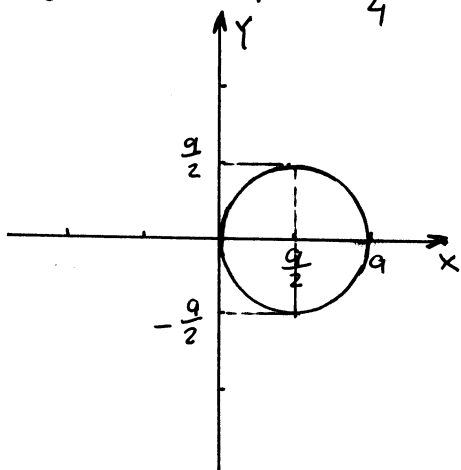
R: $x^2 + y^2 = ax$

$$x^2 - ax + y^2 = 0$$

$$x^2 - 2 \cdot x \cdot \frac{a}{2} + \frac{a^2}{4} - \frac{a^2}{4} + y^2 = 0$$

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$$

kružica centrom u $C\left(\frac{a}{2}, 0\right)$ poluprečnik $r = \frac{a}{2}$



Kako se računa krivoliniski integral $\int_L f(x,y) ds$?

Ako je kriva L data u obliku f -je $y = \eta(x)$ gdje je $a \leq x \leq b$ tada

$$\int_L f(x,y) ds = \int_a^b f(x, \eta(x)) \sqrt{1 + (\eta'(x))^2} dx$$

Ako je kriva L data u parametarskom obliku

$$\begin{cases} x = \mu(t) \\ y = \nu(t) \\ t_1 \leq t \leq t_2 \end{cases} \quad \text{tada} \quad \int_L f(x,y) ds = \int_{t_1}^{t_2} f(\mu(t), \nu(t)) \sqrt{\mu'(t)^2 + \nu'(t)^2} dt$$

Prisjetimo se polarnih koordinata

$$\begin{aligned} x &= r \cos \varphi \\ y &= r \sin \varphi \end{aligned}$$

Ako pomjerimo centar u x -osi za $\frac{a}{2}$ i fiksiramo r na $\frac{a}{2}$ imamo da je

$$L = \begin{cases} x = \frac{a}{2} + \frac{a}{2} \cos \varphi \\ y = \frac{a}{2} \sin \varphi \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

$$x' = -\frac{a}{2} \sin \varphi$$

$$y' = \frac{a}{2} \cos \varphi$$

$$(x')^2 + (y')^2 =$$

$$= \frac{a^2}{4} \sin^2 \varphi + \frac{a^2}{4} \cos^2 \varphi$$

$$= \frac{a^2}{4}$$

$$\sqrt{x'^2 + y'^2} = \frac{a}{2}$$

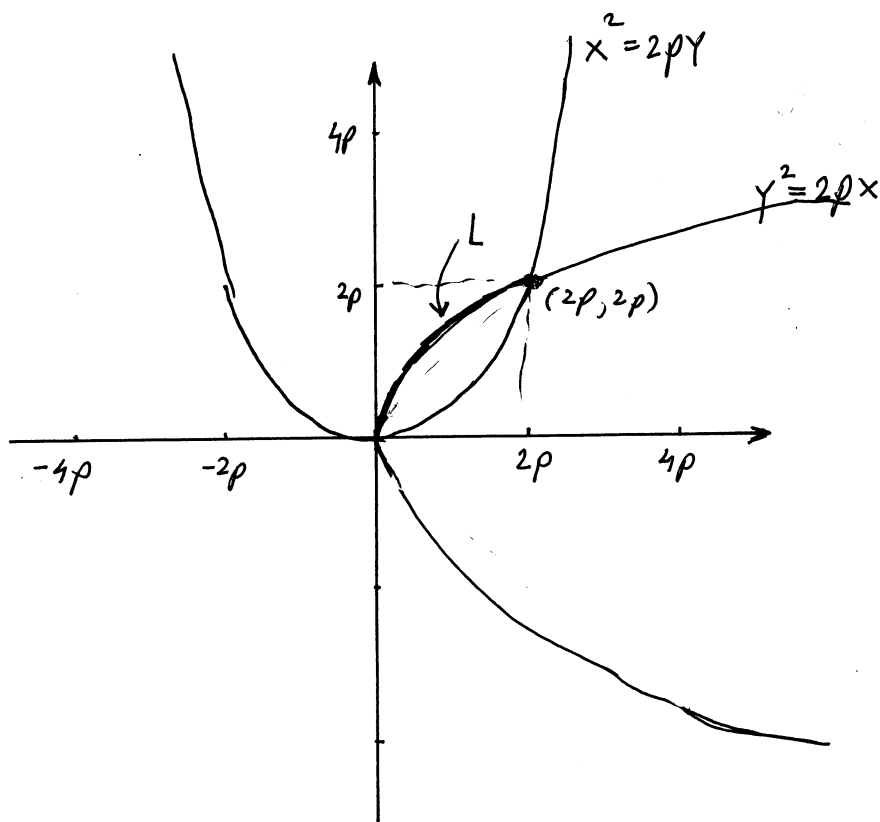
$$\int_L (x-y) ds = \int_0^{2\pi} \left(\frac{a}{2} + \frac{a}{2} \cos \varphi - \frac{a}{2} \sin \varphi\right) \cdot \frac{a}{2} d\varphi = \int_0^{2\pi} \left(\frac{a^2}{4} + \frac{a^2}{4} \cos \varphi - \frac{a^2}{4} \sin \varphi\right) d\varphi =$$

$$= \frac{a^2}{4} \varphi \Big|_0^{2\pi} + \frac{a^2}{4} \sin \varphi \Big|_0^{2\pi} + \frac{a^2}{4} \cos \varphi \Big|_0^{2\pi} = \frac{a^2}{4} \cdot 2\pi = \frac{a^2 \pi}{2}$$

traženo
rješenje

⊕ Izračunati krivolinijski integral $\int y ds$ pri čemu je L luk parabole $y^2 = 2px$, koji leži unutar parabole $x^2 = 2py$.

Rj. Skicirajmo dužice date parabole



kako se računa
Prisjetimo se krivolinijski
integral $\int_L f(x,y) ds$.

Ako je L kriva u ravni
opisana jednačinom
 $y = \eta(x)$, $a \leq x \leq b$ se računa

$$\int_L f(x,y) ds = \int_a^b f(x, \eta(x)) \sqrt{1 + (\eta'(x))^2} dx$$

U našem slučaju

$$y = \sqrt{2px} \text{ gdje je } 0 \leq x \leq 2p$$

$$y = \sqrt{2px} = \sqrt{2p} \cdot \sqrt{x}$$

$$(y')^2 = \frac{2p}{4x} = \frac{p}{2x}$$

$$y' = \sqrt{2p} \cdot \frac{1}{2\sqrt{x}} = \frac{p}{\sqrt{2px}}$$

$$1 + (y')^2 = 1 + \frac{p}{2x} = \frac{2x+p}{2x}$$

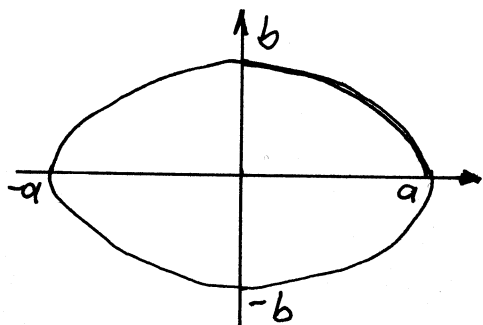
$$\int_L y ds = \int_0^{2p} \sqrt{2p} \cdot \sqrt{x} \cdot \frac{\sqrt{2x+p}}{\sqrt{2x}} dx = \sqrt{2} \cdot \sqrt{p} \cdot \frac{1}{\sqrt{2}} \int_0^{2p} \sqrt{x} \frac{\sqrt{2x+p}}{\sqrt{x}} dx = \left| \frac{d(2x+p)}{2} = dx \right|$$

$$= \sqrt{p} \int_0^{2p} \sqrt{2x+p} \cdot \frac{1}{2} d(2x+p) = \frac{\sqrt{p}}{2} \cdot \frac{2}{3} (2x+p)^{\frac{3}{2}} \Big|_0^{2p} = \frac{p^{\frac{1}{2}}}{3} \left((5p)^{\frac{3}{2}} - p^{\frac{3}{2}} \right)$$

$$= \frac{1}{3} \cdot p^{\frac{1}{2}} \cdot p^{\frac{3}{2}} (\sqrt{5^3} - 1) = \frac{p^2}{3} (5\sqrt{5} - 1) \text{ traženo rješenje}$$

#) Izračunati $\int xy ds$ gdje je c četvrtina elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ koja leži u prvom kvadrantu.

Rj. I način:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2 - \frac{b^2}{a^2} x^2$$

$$y = \pm \sqrt{b^2 \left(1 - \frac{x^2}{a^2}\right)}$$

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

$$c: \begin{cases} y = \frac{b}{a} \sqrt{a^2 - x^2} \\ 0 \leq x \leq a \end{cases}$$

$$y' = \frac{b}{a} \cdot \frac{-2x}{2\sqrt{a^2 - x^2}}$$

$$\int_c xy ds = \int_0^a x \frac{b}{a} \sqrt{a^2 - x^2} \sqrt{1 + \left(\frac{b}{a} \frac{-x}{\sqrt{a^2 - x^2}}\right)^2} dx$$

= ...

II način

Uvodimo poopštene polarne koordinate

$$x = ar \cos \varphi$$

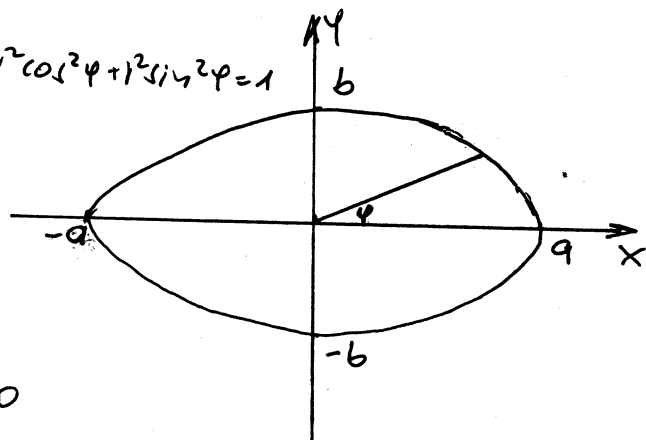
$$y = br \sin \varphi$$

$$x^2 = a^2 r^2 \cos^2 \varphi$$

$$y^2 = b^2 r^2 \sin^2 \varphi$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = r^2 \cos^2 \varphi + r^2 \sin^2 \varphi = 1$$

za $\varphi = 0$ imamo $x = a, y = 0$
za $\varphi = \frac{\pi}{2}$ imamo $x = 0, y = b$



Sad elipsu možemo napisati u parametarskom obliku tj. imamo

$$c: \begin{cases} x = a \cos \varphi \\ y = b \sin \varphi \\ 0 \leq \varphi \leq \frac{\pi}{2} \end{cases}$$

$$\frac{\partial x}{\partial \varphi} = -a \sin \varphi$$

$$\frac{\partial y}{\partial \varphi} = b \cos \varphi$$

$$\int_c f(x, y) ds = \int_a^b \int_{\alpha}^{\beta} f(\varphi(t), \psi(t)) \sqrt{\left(\frac{\partial \varphi}{\partial t}\right)^2 + \left(\frac{\partial \psi}{\partial t}\right)^2} dt \quad \text{gdje } \varphi = \varphi(t), \psi = \psi(t), \alpha \leq t \leq \beta$$

$$\int_c xy ds = \int_0^{\frac{\pi}{2}} (a \cos \varphi)(b \sin \varphi) \sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi} d\varphi = ab \int_0^{\frac{\pi}{2}} \cos \varphi \sin \varphi \sqrt{a^2 + (b^2 - a^2) \cos^2 \varphi} d\varphi$$

$$= \int_{a^2}^{b^2} \frac{-ab}{2(b^2 - a^2)} \frac{1}{2} u^{\frac{3}{2}} du = \frac{-ab}{2(b^2 - a^2)} \cdot \frac{1}{3} \frac{(a^3 - b^3)}{(a-b)(a^2 + ab + b^2)} = \frac{ab}{3(a+b)} (a^2 + ab + b^2)$$

$$= \frac{-ab}{2(b^2 - a^2)} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_{b^2}^{a^2} = \frac{-ab}{(b-a)(b+a)} \cdot \frac{1}{3} \frac{(a^3 - b^3)}{(a-b)(a^2 + ab + b^2)} = \frac{ab}{3(a+b)} (a^2 + ab + b^2)$$

⊕ Izračunati krivolinijski integral $I = \int_C z \sqrt{x^2 + y^2 + 2z^2} dS$

ako je C kriva $x = \frac{r\sqrt{2}}{2} \cos t$,
 $y = \frac{r\sqrt{2}}{2} \sin t$, $z = r \sin t$, $t \in [0, \pi]$.

Rj. Ako je C kriva opisana parametarskim jednačinama

$$C: \begin{cases} x = \mu(t) \\ y = \eta(t) \\ z = \psi(t) \end{cases}, t_1 \leq t \leq t_2 \quad \text{tada}$$

$$\int_C f(x, y, z) dS = \int_{t_1}^{t_2} f(\mu(t), \eta(t), \psi(t)) \underbrace{\sqrt{(\mu'(t))^2 + (\eta'(t))^2 + (\psi'(t))^2}}_{dS} dt$$

$$x^2 + y^2 + 2z^2 = \frac{2r^2}{4} \cos^2 t + \frac{2r^2}{4} \sin^2 t + 2r^2 \sin^2 t = r^2 \cos^2 t + 2r^2 \sin^2 t$$

$$x'_t = -\frac{\sqrt{2}}{2} r \sin t, \quad y'_t = \frac{\sqrt{2}}{2} r \cos t, \quad z'_t = r \cos t$$

$$(x'_t)^2 + (y'_t)^2 + (z'_t)^2 = \frac{1}{2} r^2 \sin^2 t + \frac{1}{2} r^2 \cos^2 t + r^2 \cos^2 t = r^2 \sin^2 t + r^2 \cos^2 t = r^2 (\sin^2 t + \cos^2 t) = r^2$$

$$\sqrt{(\mu'(t))^2 + (\eta'(t))^2 + (\psi'(t))^2} = \sqrt{r^2} = r$$

$$I = \int_C z \sqrt{x^2 + y^2 + 2z^2} dS = \int_0^\pi r \sin t \sqrt{\frac{r^2 \cos^2 t + 2r^2 \sin^2 t}{r^2 (\cos^2 t + 2\sin^2 t)}} r dt =$$

$$= r^3 \int_0^\pi \sin t \sqrt{\frac{\cos^2 t + 2\sin^2 t}{1 - \cos^2 t}} dt = r^3 \int_0^\pi \sin t \sqrt{2 - \cos^2 t} dt =$$

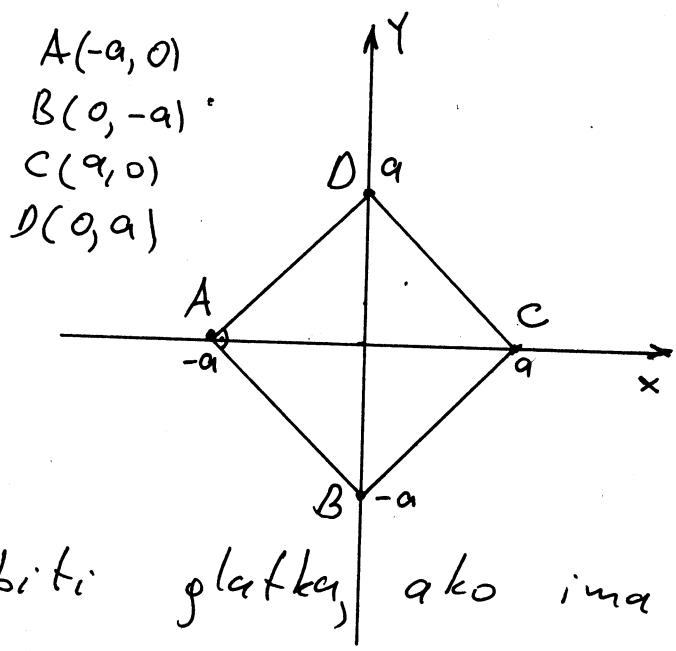
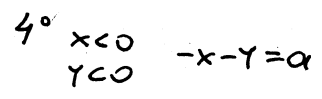
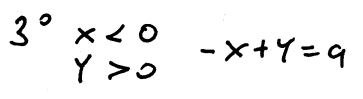
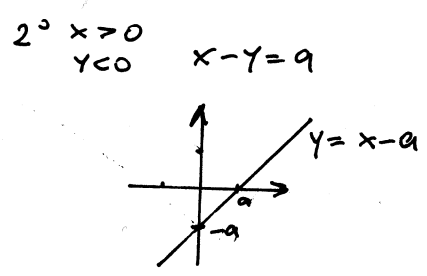
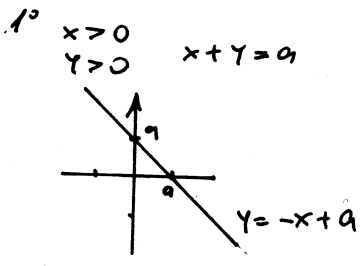
$$= \left. \begin{array}{l} \cos t = u \\ -\sin t dt = du \\ \sin t dt = -du \end{array} \right|_{t=0}^{t=\pi} \Rightarrow u \Big|_1^{-1} = r^3 \int_{-1}^1 \sqrt{2 - t^2} dt = r^3 \int_{-1}^1 \frac{2 - t^2}{\sqrt{2 - t^2}} dt$$

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VJE BV

$$\dots = r^3 \cdot \frac{1}{2} t \sqrt{2 - t^2} \Big|_{-1}^1 + r^3 \int_{-1}^1 \frac{dt}{2 - t^2} = \dots = \left(1 + \frac{\pi}{2}\right) r^3$$

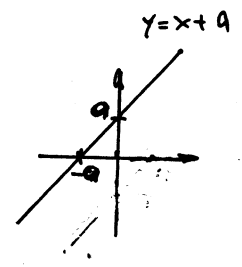
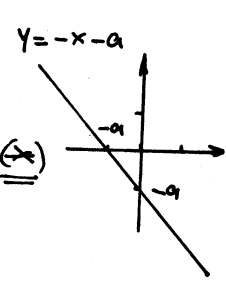
Izračunati: integral po krivoj $C \int_C xy \, ds$ gdje je C kvadrat $|x|+|y|=a$, $a>0$.

R: j) Kako nacrtati kvadrat $|x|+|y|=a$?



Kriva po kojoj se integrirati mora biti glatka, ako ima čošak razbije se na dijelove.

$$\int_C xy \, ds = \int_{AB} xy \, ds + \int_{BC} xy \, ds + \int_{CD} xy \, ds + \int_{DA} xy \, ds$$

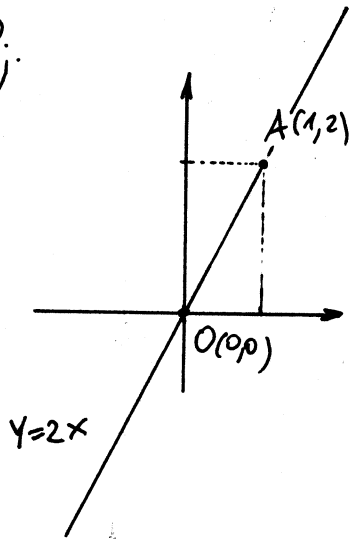


$$\int_C f(x, y) \, ds = \int_a^b f(x, \varphi(x)) \sqrt{1 + (\varphi'(x))^2} \, dx, \text{ gdje je } \varphi(x) \text{ kriva } x \in [a, b]$$

$$\begin{aligned}
 & \int_{-a}^0 x(-x-a) \sqrt{1+(-1)^2} \, dx + \int_0^a x(x-a) \sqrt{1+1^2} \, dx + \int_0^a x(-x+a) \sqrt{1+(-1)^2} \, dx \\
 & + \int_{-a}^0 x(x+a) \sqrt{1+1^2} \, dx = \sqrt{2} \left(\int_{-a}^0 (-x^2 - ax + x^2 + ax) \, dx + \int_0^a (x^2 - ax - x^2 + ax) \, dx \right) = 0
 \end{aligned}$$

Izračunati integral $\int \frac{ds}{\sqrt{x^2+y^2+4}}$ gdje je c duž koja spaja tačke $O(0,0)$ i tačku $A(1,2)$.

Rj.



$c: y=2x$

$y' = 2$

$$\int_c f(x,y) ds = \int_a^b f(x, \varphi(x)) \sqrt{1+(\varphi'(x))^2} dx$$

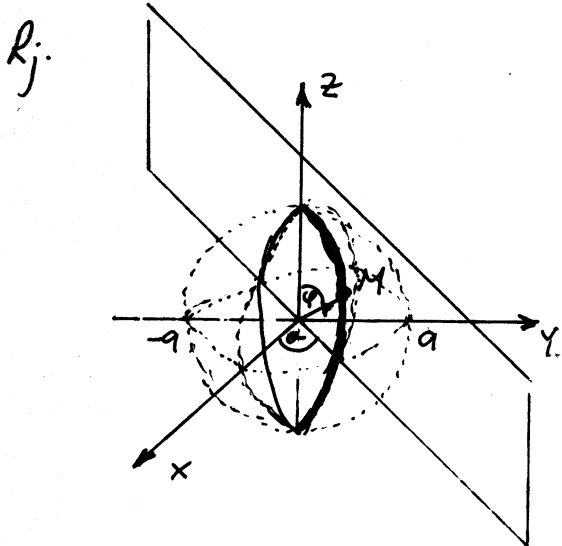
gdje je $y = \varphi(x)$. kriva $x \in [a, b]$

$$\int_c \frac{1}{\sqrt{x^2+y^2+4}} ds = \int_0^1 \frac{\sqrt{1+2^2}}{\sqrt{x^2+(2x)^2+4}} dx = \sqrt{5} \int_0^1 \frac{dx}{\sqrt{5x^2+4}}$$

$$= \sqrt{5} \int_0^1 \frac{dx}{\sqrt{4(\frac{5}{4}x^2+1)}} = \frac{\sqrt{5}}{2} \int_0^1 \frac{d(\frac{\sqrt{5}}{2}x)}{\sqrt{(\frac{\sqrt{5}}{2}x)^2+1}} \cdot \frac{2}{\sqrt{5}} = \ln \left| \frac{\sqrt{5}x}{2} + \sqrt{\left(\frac{\sqrt{5}x}{2}\right)^2+1} \right| \Big|_0^1$$

$$= \ln \left| \frac{\sqrt{5}}{2} + \sqrt{\frac{5}{4}+1} \right| - \ln 1 = \ln \left| \frac{\sqrt{5}}{2} + \sqrt{\frac{9}{4}} \right| = \ln \frac{\sqrt{5}+3}{2}$$

Izračunati $\int \sqrt{2y^2 + z^2} ds$ gdje je c krug dobijen presjekom sfere $x^2 + y^2 + z^2 = a^2$ i ravni $x = y$.



Kako ćemo opisati ^{datu} sferu parametarstki? (sferne koordinate)

$$x = r \sin \varphi \cos \alpha$$

$$y = r \sin \varphi \sin \alpha$$

$$z = r \cos \varphi$$

$$r = a$$

$$0 \leq \alpha \leq 2\pi$$

$$0 \leq \varphi \leq \pi$$

Kako da parametarstki opišemo krug dobijen presjekom sfere i ravni?

Za pravu $x = y$ znamo da je ugao između ove prave i x -ose 45° . Prema tome $\alpha = 45^\circ$, ($r = a$):

$$c: \begin{cases} x = \frac{\sqrt{2}}{2} a \sin \varphi \\ y = \frac{\sqrt{2}}{2} a \sin \varphi \\ z = a \cos \varphi \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

a, r su
fiksirani

$$2y^2 + z^2 = 2 \cdot \frac{2}{4} a^2 \sin^2 \varphi + a^2 \cos^2 \varphi = a^2$$

Ako je kriva c opisana sa $x = \mu(t)$, $y = \eta(t)$, $z = \xi(t)$, $a < t < b$ onda je

$$\int_c f(x, y, z) ds = \int_a^b f(\mu(t), \eta(t), \xi(t)) \sqrt{(\mu'(t))^2 + (\eta'(t))^2 + (\xi'(t))^2} dt$$

$$\frac{\partial x}{\partial \varphi} = \frac{\sqrt{2}}{2} a \cos \varphi$$

$$\frac{\partial y}{\partial \varphi} = \frac{\sqrt{2}}{2} a \cos \varphi$$

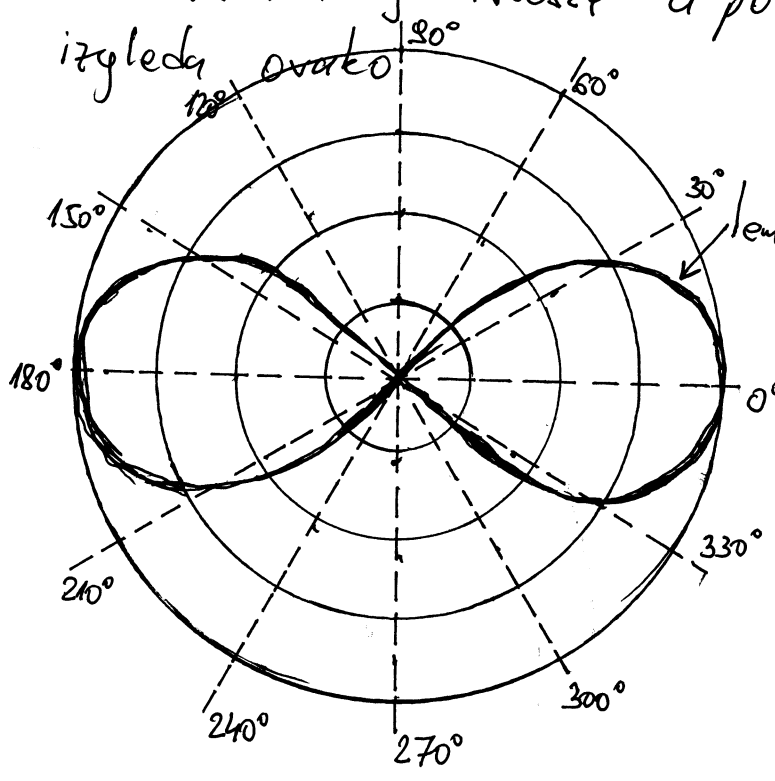
$$\frac{\partial z}{\partial \varphi} = -a \sin \varphi$$

$$\int_c \sqrt{2y^2 + z^2} ds = \int_0^{2\pi} \sqrt{a^2} \cdot \sqrt{\frac{2}{4} a^2 \cos^2 \varphi + \frac{2}{4} a^2 \cos^2 \varphi + a^2 \sin^2 \varphi} dt$$

$$= \int_0^{2\pi} a \cdot \sqrt{a^2 (\cos^2 \varphi + \sin^2 \varphi)} dt = a^2 \int_0^{2\pi} dt = 2a^2 \pi$$

Izračunati krivolinijski integral prve vrste $\int (x+y) dS$, ako je c desna latica lemniskate $\rho = a\sqrt{\cos 2\varphi}$.

R.j. Lemniskata $\rho = a\sqrt{\cos 2\varphi}$ u polarnom koordinatnom sistemu izgleda ovako



Data kriva je prikazana u polarnim koordinatama

$$c: \begin{cases} \rho = a\sqrt{\cos 2\varphi} \\ \varphi \in [-\frac{\pi}{4}, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}] \end{cases}$$

Prezjetimo se,

$$\int_c (x+y) dS = \int_{t_1}^{t_2} (\eta(t) + \mu(t)) \sqrt{(\eta'(t))^2 + (\mu'(t))^2} dt$$

ako je c data u obliku

$$c: \begin{cases} x = \eta(t) \\ y = \mu(t) \\ t_1 \leq t \leq t_2 \end{cases}$$

kao pomoć uvedimo polarne koordinate

$$\int_c (x+y) dS = \begin{cases} x = \rho \cos \varphi \\ y = \rho \sin \varphi \\ \text{za } \rho \text{ dano a zebit} \\ \rho = a\sqrt{\cos 2\varphi} \end{cases}$$

Prava točka desna latica lemniskate

$$c: \begin{cases} x = a \cos \varphi \sqrt{\cos 2\varphi} \\ y = a \sin \varphi \sqrt{\cos 2\varphi} \\ -\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4} \end{cases}$$

$$x' = (a(-\sin \varphi) \sqrt{\cos 2\varphi} + a \cos \varphi \cdot \frac{1}{2} (\cos 2\varphi)^{-\frac{1}{2}} \cdot (-\sin 2\varphi) \cdot 2) d\varphi = a(-\sin \varphi \sqrt{\cos 2\varphi} - \cos \varphi \frac{\sin 2\varphi}{\sqrt{\cos 2\varphi}}) d\varphi$$

$$y' = (a \cos \varphi \sqrt{\cos 2\varphi} + a \sin \varphi \cdot \frac{1}{2} (\cos 2\varphi)^{-\frac{1}{2}} \cdot (-\sin 2\varphi) \cdot 2) d\varphi = (a \cos \varphi \sqrt{\cos 2\varphi} - a \sin \varphi \frac{\sin 2\varphi}{\sqrt{\cos 2\varphi}}) d\varphi$$

zato što posmatramo desnu stranu lemniskate

adiciona teor. ... $= a \frac{\cos 3\varphi}{\sqrt{\cos 2\varphi}} d\varphi$

ad. teor. ... $= -a \frac{\sin 3\varphi}{\sqrt{\cos 2\varphi}} d\varphi$

$$x'^2 + y'^2 = a^2 \frac{\sin^2 3\varphi}{\cos 2\varphi} d\varphi^2 + a^2 \frac{\cos^2 3\varphi}{\cos 2\varphi} d\varphi^2 = a^2 \frac{1}{\cos 2\varphi} d\varphi^2$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\cos 2\varphi} a (\cos \varphi + \sin \varphi) \cdot a \frac{1}{\sqrt{\cos 2\varphi}} d\varphi = a^2 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos \varphi + \sin \varphi) d\varphi =$$

$$= a^2 \left(\sin \varphi \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \cos \varphi \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \right) = a^2 \sqrt{2} \text{ traženo rješenje.}$$

Zadaci za vježbu

U zadacima 3770—3775 izračunati date krivolinijske integrale.

3770. $\int_L \frac{ds}{x-y}$, pri čemu je L odsečak na pravoj $y = \frac{1}{2}x - 2$, koji leži

između tačaka $A(0, -2)$ i $B(4, 0)$.

3771. $\int_L xy ds$, pri čemu je L kontura pravougaonika čija su temena

$A(0, 0)$, $B(4, 0)$, $C(4, 2)$ i $D(0, 2)$.

3772. $\int_L y ds$, pri čemu je L luk parabole $y^2 = 2px$, koji leži unutar

parabole $x^2 = 2py$.

3773. $\int_L (x^2 + y^2)^n ds$, pri čemu je L krug $x = a \cos t$, $y = a \sin t$.

3774. $\int_L xy ds$, pri čemu je L četvrtina elipse koja leži u prvom kva-

drantu.

3775. $\int_L \sqrt{2y} ds$, pri čemu je L prvi svod cikloide $x = a(t - \sin t)$,

$$y = a(1 - \cos t).$$

3776. Napisati obrazac za izračunavanje integrala $\int_L F(x, y) ds$ u po-

larnim koordinatama, ako je kriva L zadata jednačinom $\rho = \rho(\varphi)$ ($\varphi_1 \leq \varphi \leq \varphi_2$).

3777*. Izračunati $\int_L (x-y) ds$, po kružnoj liniji $x^2 + y^2 = ax$.

3778. Izračunati $\int_L \sqrt{x^2 - y^2} ds$ po krivoj $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ ($x \geq 0$) (polo-

vina lemniskate).

3779. Izračunati $\int_L \arctg \frac{y}{x} ds$ po delu Arhimedove spirale $\rho = 2\varphi$ koji

leži unutar kruga poluprečnika R , čiji je centar u koordinatnom početku.

3780. Izračunati $\int_L \frac{z^2 ds}{x^2 + y^2}$ po prvom zavoju zavojnice $x = a \cos t$, $y = a \sin t$,

$$z = at.$$

3781. Izračunati $\int_L xyz ds$ po delu kružne linije $x^2 + y^2 + z^2 = R^2$,

$$x^2 + y^2 = \frac{R^2}{4}, \text{ koji leži u prvom oktantu.}$$

3782. Izračunati $\int_L (2z - \sqrt{x^2 + y^2}) ds$ po prvom zavoju konusne zavojnice

$$x = t \cos t, y = t \sin t, z = t.$$

3783. Izračunati $\int_L (x+y) ds$ po delu kružne linije $x^2 + y^2 + z^2 = R^2$, $y = x$,

koji leži u prvom oktantu.

Rješenja

3770. $\sqrt{5} \ln 2$. 3771. 24.

3772. $\frac{p^2}{3}(5\sqrt{5}-1)$. 3773. $2\pi a^{2n+1}$.

3774. $\frac{ab(a^2 + ab + b^2)}{3(a+b)}$. 3775. $4\pi a\sqrt{a}$.

3776. $\int_{\varphi_1}^{\varphi_2} F(\rho \cos \varphi, \rho \sin \varphi) \sqrt{\rho^2 + \rho'^2} d\varphi$.

3777*. $\frac{\pi a^2}{2}$. Preći na polarne koordinate.

3778. $\frac{2a^3\sqrt{2}}{3}$. 3779. $\frac{1}{12}[(R^2+4)^{\frac{3}{2}}-8]$.

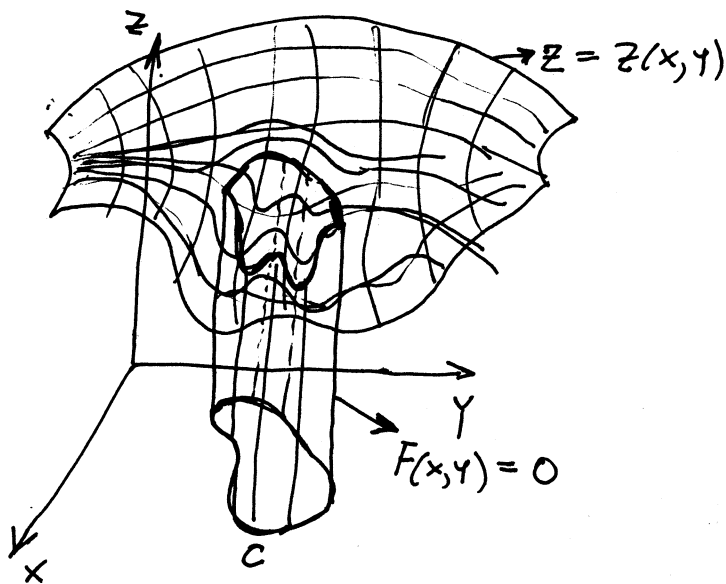
3780. $\frac{8\pi a^2\sqrt{2}}{3}$. 3781. $\frac{R^4\sqrt{3}}{32}$.

3782. $\frac{2\sqrt{2}}{3}[(1+2\pi^2)^{\frac{3}{2}}-1]$. 3783. $R^2\sqrt{2}$.

Računanje površine cilindrične površi

Ako je S dio cilindrične površine $F(x, y) = 0$ između xOy ravni i neke površine $z = z(x, y)$ tada se površina $P(S)$ površi S računa po formuli:

$$P(S) = \int_C z(x, y) dS \quad \text{gdje je} \quad C: \begin{cases} F(x, y) = 0 \\ z = 0 \end{cases}$$



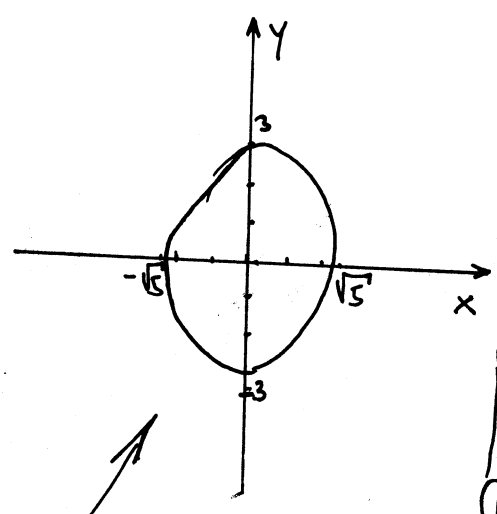
$P(S)$ - površina dijela
cilindrične
površni

Izračunati površinu eliptičkog valjka $9x^2 + 5y^2 = 45$ koji se nalazi između površi $z=0$ i $z=y$.

Rj. $P(S) = \int_C z(x,y) dS$ gdje je $C: \begin{cases} F(x,y) = 0 \\ z = 0 \end{cases}$

Skicirajmo valjak $9x^2 + 5y^2 = 45$ i: 45 u xOy ravni on izgleda

$z=0$ je xOy ravan
 $z=y$ u yOz ravni

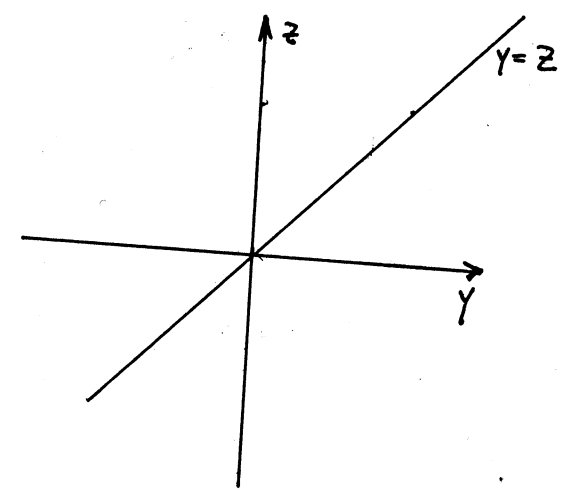


$$\frac{x^2}{5} + \frac{y^2}{9} = 1$$

elipsa

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\sqrt{5}$ i 3



$z(x,y) = y$

$C: \begin{cases} 9x^2 + 5y^2 = 45 \\ z = 0 \end{cases}$

C je elipsa

$z=y$ je ravan koja sadrži $x=0$ i yOz ravan sadrži $y=z$ pravu

Svedimo elipsu $\frac{x^2}{5} + \frac{y^2}{9} = 1$ na parametarski oblik

$x = a \cos t$
 $y = b \sin t$

U našem slučaju $x = \sqrt{5} \cos t$
 $y = 3 \sin t$
 $0 \leq t \leq 2\pi$

$C: \begin{cases} x = \mu(t) \\ y = \eta(t) \\ t_1 \leq t \leq t_2 \end{cases}$ $\int_C f(x,y) dS = \int_{t_1}^{t_2} f(\mu(t), \eta(t)) \sqrt{(\mu'(t))^2 + (\eta'(t))^2} dt$

$dS = \sqrt{5 \sin^2 t + 9 \cos^2 t} dt$ Kako se ravni $z=0$ i $z=y$ sijeku u $x=0$, to će parametar t uzimati vrijednosti od 0 do π

$P(S) = \int_C y dS = \int_0^\pi 3 \sin t \sqrt{5 \sin^2 t + 9 \cos^2 t} dt = 3 \int_0^\pi \sin t \sqrt{5(1 - \cos^2 t) + 9 \cos^2 t} dt =$

$= 3 \int_0^\pi \sin t \sqrt{5 + 4 \cos^2 t} dt = \left| \begin{matrix} 2 \cos t = u \\ -2 \sin t dt = du \\ \sin t dt = -\frac{1}{2} u \end{matrix} \right|_{t=0}^{t=\pi} = 3 \int_{u=2}^{u=-2} (-\frac{1}{2}) \sqrt{5+u^2} du =$

$= 3 \cdot \frac{1}{2} \cdot 2 \int_0^2 \sqrt{5+u^2} du = 3 \int_0^2 \frac{5+u^2}{\sqrt{5+u^2}} du = 3 \int_0^2 \frac{5}{\sqrt{5+u^2}} du + 3 \int_0^2 \frac{u^2}{\sqrt{5+u^2}} du = \left| \begin{matrix} u=x \\ ZAVRŠITI SAMI \dots \end{matrix} \right| = \frac{15\sqrt{5}}{4} + \dots$

Izračunati površinu dijela valjka $x^2 + y^2 = 1$ koji se nalazi između površi $z=0$ i $z = \sqrt{x^2 + y^2} + \sqrt{1-x^2} + \sqrt{1-y^2}$

R: $P(S) = \int_C z(x, y) dS$ gdje je $C: \begin{cases} F(x, y) = 0 \\ z = 0 \end{cases}$

U ovom slučaju je $z(x, y) = \sqrt{x^2 + y^2} + \sqrt{1-x^2} + \sqrt{1-y^2}$

$C: \begin{cases} x^2 + y^2 = 1 \\ z = 0 \end{cases}$ tj. $C: x^2 + y^2 = 1$

Parametarizirajmo kružnicu: $\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \\ 0 \leq \varphi \leq 2\pi \end{cases}$

U našem slučaju: $\begin{cases} x = \cos \varphi \\ y = \sin \varphi \\ 0 \leq \varphi \leq 2\pi \end{cases}$

$C: \begin{cases} x = \mu(t) \\ y = \eta(t) \\ t_1 \leq t \leq t_2 \end{cases}$ $\int_C f(x, y) dS = \int_{t_1}^{t_2} f(\mu(t), \eta(t)) \sqrt{(\mu'(t))^2 + (\eta'(t))^2} dt$

$(\cos \varphi)' = -\sin \varphi$

$(\sin \varphi)' = \cos \varphi$

$dS = \sqrt{\sin^2 \varphi + \cos^2 \varphi} d\varphi = d\varphi$

$\begin{cases} \sqrt{x^2 + y^2} = 1 \\ \sqrt{1-x^2} = \cos \varphi \\ \sqrt{1-y^2} = \sin \varphi \end{cases}$

Definiciono područje f-je $z = \sqrt{x^2 + y^2} + \sqrt{1-x^2} + \sqrt{1-y^2}$ je

$\{(x, y) | -1 \leq x \leq 1 \text{ i } -1 \leq y \leq 1\}$ zlo simetričnosti četini dijela $\downarrow \frac{\pi}{2}$

$P(S) = \int_C (\sqrt{x^2 + y^2} + \sqrt{1-x^2} + \sqrt{1-y^2}) dS = 4 \int_0^{\frac{\pi}{2}} (1 + \sin \varphi + \cos \varphi) d\varphi =$
 $= 4 \left[\varphi \Big|_0^{\frac{\pi}{2}} - \cos \varphi \Big|_0^{\frac{\pi}{2}} + \sin \varphi \Big|_0^{\frac{\pi}{2}} \right] = 4 \left(\frac{\pi}{2} + 1 + 1 \right) = 2\pi + 8$

Izračunati površinu cilindra $x^2 + y^2 = R^2$ između ravni $z=0$ i površi $z = R + \frac{x^2}{R}$

Zadaci za vježbu

U zadacima 3792 — 3797 izračunati površine datih cilindričnih omotača, koji leže između ravni Oxy i navedenih površina.

$$3792. \quad x^2 + y^2 = R^2, \quad z = R + \frac{x^2}{R}.$$

$$3793. \quad y^2 = 2px, \quad z = \sqrt{2px - 4x^2}.$$

$$3794. \quad y^2 = \frac{4}{9}(x-1)^3, \quad z = 2 - \sqrt{x}.$$

$$3795. \quad x^2 + y^2 = R^2, \quad 2Rz = xy.$$

$$3796. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = kx \text{ i } z = 0 \quad (z \geq 0) \quad (\text{„cilindrična potkovica“})$$

$$3797. \quad y = \sqrt{2px}, \quad z = y \text{ i } x = \frac{8}{9}p.$$

3798. Izračunati površinu onog dela kružnog cilindra koji iz njega iseca drugi isti takav cilindar, ako im se ose seku pod pravim uglom a poluprečnici su im R (uporedi sa rešenjem zadatka 3642).

3799. Naći površinu onog dela cilindra $x^2 + y^2 = R^2$, koji leži unutar sfere $x^2 + y^2 + z^2 = R^2$.

Rješenja

$$3792. \quad 3\pi R^2. \quad 3793. \quad \frac{\pi p^2}{4}. \quad 3794. \quad \frac{11}{3}. \quad 3795. \quad R^2.$$

$$3796. \quad ka \left(a + \frac{b^2}{2c} \ln \frac{a+c}{a-c} \right), \text{ gde je } c = \sqrt{a^2 - b^2}. \text{ Za } a-b \leq S = 2ka^2.$$

$$3797. \quad \frac{98}{81}p^2. \quad 3798. \quad 8R^2. \quad 3799. \quad 4R^2.$$